

# Very well-covered graphs and the unimodality conjecture

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February 1, 2008

## Abstract

If  $s_k$  denotes the number of stable sets of cardinality  $k$  in the graph  $G$ , then  $I(G; x) = \sum_{k=0}^{\alpha(G)} s_k x^k$  is the *independence polynomial* of  $G$  (Gutman, Harary, 1983), where  $\alpha(G)$  is the size of a maximum stable set in  $G$ . Alavi, Malde, Schwenk and Erdős (1987) conjectured that  $I(T, x)$  is unimodal for any tree  $T$ , while, in general, they proved that for any permutation  $\pi$  of  $\{1, 2, \dots, \alpha\}$  there is a graph  $G$  with  $\alpha(G) = \alpha$  such that  $s_{\pi(1)} < s_{\pi(2)} < \dots < s_{\pi(\alpha)}$ . Brown, Dilcher and Nowakowski (2000) conjectured that  $I(G; x)$  is unimodal for any well-covered graph. Michael and Traves (2002) provided examples of well-covered graphs with non-unimodal independence polynomials. They proposed the so-called "roller-coaster" conjecture: for a well-covered graph, the subsequence  $(s_{\lceil \alpha/2 \rceil}, s_{\lceil \alpha/2 \rceil+1}, \dots, s_\alpha)$  is unconstrained in the sense of Alavi *et al.* The conjecture of Brown *et al.* is still open for very well-covered graphs, and it is worth mentioning that, apart from  $K_1$  and the chordless cycle  $C_7$ , connected well-covered graphs of girth  $\geq 6$  are very well-covered (Finbow, Hartnell and Nowakowski, 1993).

In this paper we prove that  $s_{\lceil (2\alpha-1)/3 \rceil} \geq \dots \geq s_{\alpha-1} \geq s_\alpha$  are valid for any (a) bipartite graph  $G$  with  $\alpha(G) = \alpha$ ; (b) quasi-regularizable graph  $G$  on  $2\alpha(G) = 2\alpha$  vertices. In particular, we infer that this is true for (a) trees, thus doing a step in an attempt to prove Alavi *et al.*' conjecture; (b) very well-covered graphs. Consequently, for this case, the unconstrained subsequence appearing in the roller-coaster conjecture can be shorten to  $(s_{\lceil \alpha/2 \rceil}, s_{\lceil \alpha/2 \rceil+1}, \dots, s_{\lceil (2\alpha-1)/3 \rceil})$ . We also show that the independence polynomial of a very well-covered graph  $G$  is unimodal for  $\alpha(G) \leq 9$ , and is log-concave whenever  $\alpha(G) \leq 5$ .

**key words:** *stable set, independence polynomial, unimodal sequence, quasi-regularizable graph, bipartite graph, very well-covered graph.*

# 1 Introduction

Throughout this paper  $G = (V, E)$  is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set  $V = V(G)$  and edge set  $E = E(G)$ . If  $X \subset V$ , then  $G[X]$  is the subgraph of  $G$  spanned by  $X$ . By  $G - W$  we mean the subgraph  $G[V - W]$ , if  $W \subset V(G)$ . We also denote by  $G - F$  the partial subgraph of  $G$  obtained by deleting the edges of  $F$ , for  $F \subset E(G)$ , and we write shortly  $G - e$ , whenever  $F = \{e\}$ .

A vertex  $v$  is *pendant* if its neighborhood  $N(v) = \{u : u \in V, uv \in E\}$  contains only one vertex; an edge  $e = uv$  is *pendant* if one of its endpoints is a pendant vertex.  $\overline{G}$  stands for the complement of  $G$ , while  $K_n, P_n, C_n$  denote respectively, the complete graph on  $n \geq 1$  vertices, the chordless path on  $n \geq 1$  vertices, and the chordless cycle on  $n \geq 3$  vertices. As usual, a *tree* is an acyclic connected graph.

A set of pairwise non-adjacent vertices is called *stable*. If  $S$  is a stable set, then we denote  $N(S) = \{v : N(v) \cap S \neq \emptyset\}$  and  $N[S] = N(S) \cup S$ . A stable set of maximum size will be referred to as a *maximum stable set* of  $G$ . The *stability number* of  $G$ , denoted by  $\alpha(G)$ , is the cardinality of a maximum stable set in  $G$ , and  $\omega(G) = \alpha(\overline{G})$ .

The *disjoint union* of the graphs  $G_1, G_2$  is the graph  $G = G_1 \sqcup G_2$  having as vertex set and edge set the disjoint unions of  $V(G_1), V(G_2)$  and  $E(G_1), E(G_2)$ , respectively.

If  $G_1, G_2$  are disjoint graphs, then their *Zykov sum*, (Zykov, [24], [25]), is the graph  $G_1 + G_2$  with

$$\begin{aligned} V(G_1 + G_2) &= V(G_1) \cup V(G_2), \\ E(G_1 + G_2) &= E(G_1) \cup E(G_2) \cup \{v_1v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}. \end{aligned}$$

In particular,  $\sqcup nG$  and  $+nG$  denote the disjoint union and Zykov sum, respectively, of  $n > 1$  copies of the graph  $G$ .

A graph  $G$  is called *quasi-regularizable* if one can replace each edge of  $G$  with a non-negative integer number of parallel copies, so as to obtain a regular multigraph of degree  $\neq 0$  (see [3], [4]). Evidently, a disconnected quasi-regularizable graph has no isolated vertices. Moreover, a disconnected graph is quasi-regularizable if and only if any of its connected components spans a quasi-regularizable graph. The following characterization of quasi-regularizable graphs, due to Berge, we shall use in the sequel.

**Theorem 1.1** [3] *A graph  $G$  is quasi-regularizable if and only if  $|S| \leq |N(S)|$  holds for any stable set  $S$  of  $G$ .*

Let  $s_k$  be the number of stable sets in  $G$  of cardinality  $k \in \{0, 1, \dots, \alpha(G)\}$ . The polynomial

$$I(G; x) = s_0 + s_1x + s_2x^2 + \dots + s_\alpha x^\alpha, \quad \alpha = \alpha(G),$$

is called the *independence polynomial* of  $G$  (Gutman and Harary, [10]). Various properties of this polynomial are presented in a number of papers, like [10], [5], [6], [12], [15], [16], [17], [18], [19], [21].

A finite sequence of real numbers  $(a_0, a_1, a_2, \dots, a_n)$  is said to be:

- *unimodal* if there is some  $k \in \{0, 1, \dots, n\}$ , called the *mode* of the sequence, such that  $a_0 \leq \dots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \dots \geq a_n$ ;
- *log-concave* if  $a_i^2 \geq a_{i-1} \cdot a_{i+1}$  holds for  $i \in \{1, 2, \dots, n-1\}$ .

It is known that any log-concave sequence of positive numbers is also unimodal.

A polynomial is called *unimodal* (*log-concave*) if the sequence of its coefficients is unimodal (log-concave, respectively). For instance, the independence polynomial  $I(K_{1,3}; x) = 1 + 4x + 3x^2 + x^3$  is unimodal. However, the independence polynomial of  $G = K_{24} + (K_3 \sqcup K_3 \sqcup K_4)$  is not unimodal, since  $I(G; x) = 1 + 34x + 33x^2 + 36x^3$  (for other examples, see [1]). Moreover, Alavi *et al.* [1] proved that for any permutation  $\pi$  of  $\{1, 2, \dots, \alpha\}$  there is a graph  $G$  with  $\alpha(G) = \alpha$  such that  $s_{\pi(1)} < s_{\pi(2)} < \dots < s_{\pi(\alpha)}$ . Nevertheless, for trees, they stated the following still open conjecture.

**Conjecture 1.2** [1] *The independence polynomial of a tree is unimodal.*

A graph  $G$  is called *well-covered* if all its maximal stable sets have the same cardinality, (Plummer, [22]). If, in addition,  $G$  has no isolated vertices and its order equals  $2\alpha(G)$ , then  $G$  is *very well-covered* (Favaron, [8]).

By  $G^*$  we mean the graph obtained from  $G$  by appending a single pendant edge to each vertex of  $G$ , [7]. Let us notice that  $G^*$  is well-covered (see, for instance, [13]), and  $\alpha(G^*) = n$ . In fact,  $G^*$  is very well-covered. Moreover, the following result shows that, under certain conditions, any well-covered graph equals  $G^*$  for some graph  $G$ .

**Theorem 1.3** [9] *Let  $G$  be a connected graph of girth  $\geq 6$ , which is isomorphic to neither  $C_7$  nor  $K_1$ . Then  $G$  is well-covered if and only if its pendant edges form a perfect matching.*

In other words, Theorem 1.3 shows that, apart from  $K_1$  and  $C_7$ , connected well-covered graphs of girth  $\geq 6$  are very well-covered. For example, a tree  $T \neq K_1$  could be only very well-covered, and this is the case if and only if  $T = G^*$  for some tree  $G$  (see also [23], [8], [14]).

In [5] it was conjectured that the independence polynomial of any well-covered graph  $G$  is unimodal. Michael and Traves [21] proved that this conjecture is true for  $\alpha(G) \in \{1, 2, 3\}$ , but it is false for  $\alpha(G) \in \{4, 5, 6, 7\}$ . A family of well-covered graphs with non-unimodal independence polynomials and stability numbers  $\geq 8$  is presented in [19]. However, the conjecture is still open for very well-covered graphs. In [15] and [16], unimodality of independence polynomials of a number of well-covered trees (e.g.,  $P_n^*$ ,  $K_{1,n}^*$ ) was validated, using the fact that the independence polynomial of a claw-free graph is unimodal (a result due to Hamidoune, [11]). We also showed that  $I(G^*; x)$  is unimodal for any  $G^*$  whose skeleton  $G$  has  $\alpha(G) \leq 4$  (see [17]).

Michael and Traves formulated (and verified for well-covered graphs with stability numbers  $\leq 7$ ) the following so-called "roller-coaster" conjecture.

**Conjecture 1.4** [21] For any permutation  $\pi$  of the set  $\{\lceil \alpha/2 \rceil, \lceil \alpha/2 \rceil + 1, \dots, \alpha\}$ , there exists a well-covered graph  $G$ , with  $\alpha(G) = \alpha$ , whose sequence  $(s_0, s_1, \dots, s_\alpha)$  satisfies  $s_{\pi(\lceil \alpha/2 \rceil)} < s_{\pi(\lceil \alpha/2 \rceil + 1)} < \dots < s_{\pi(\alpha)}$ .

In this paper we prove that if  $G$  is a quasi-regularizable graph on  $2\alpha(G)$  vertices, then  $s_{\lceil (2\alpha(G)-1)/3 \rceil} \geq s_{\lceil (2\alpha(G)-1)/3 \rceil + 1} \geq \dots \geq s_{\alpha(G)}$ , while if  $G$  is a perfect graph, then  $s_{\lceil (\omega\alpha-1)/(\omega+1) \rceil} \geq s_{\lceil (\omega\alpha-1)/(\omega+1) \rceil + 1} \geq \dots \geq s_\alpha$ , where  $\alpha = \alpha(G)$ ,  $\omega = \omega(G)$ . We infer that for very well-covered graphs, the domain of the roller-coaster conjecture can be shorten to  $\{\lceil \alpha/2 \rceil, \lceil \alpha/2 \rceil + 1, \dots, \lceil (2\alpha-1)/3 \rceil\}$ . Moreover, we show that the independence polynomial of a very well-covered graph  $G$  is unimodal for  $\alpha(G) \leq 9$ , and log-concave, whenever  $\alpha(G) \leq 5$ .

## 2 Results

In [5] it was shown that  $s_{k-1} \leq k \cdot s_k$  and  $s_k \leq (n - k + 1) \cdot s_{k-1}$ ,  $1 \leq k \leq \alpha(G)$ , are true for any well-covered graph  $G$  on  $n$  vertices.

**Proposition 2.1** [21], [18] If  $G$  is a well-covered graph with  $\alpha(G) = \alpha$ , then the following statements are true:

- (i)  $(\alpha - k) \cdot s_k \leq (k + 1) \cdot s_{k+1}$  holds for  $0 \leq k < \alpha$ ;
- (ii)  $s_{k-1} \leq s_k$  for any  $1 \leq k \leq (\alpha + 1)/2$ .

Notice that Proposition 2.1(i) can fail for non-well-covered graphs, e.g., the graph  $G_1$  in Figure 1 has  $\alpha(G_1) = 3$  and  $(\alpha(G_1) - 2) \cdot s_2 = 8 > 3 = (2 + 1) \cdot s_3$ . However, there are non-well-covered graphs satisfying Proposition 2.1(i), for instance, the graph  $G_2$  in Figure 1. Since  $I(G_1; x) = 1 + 6x + 8x^2 + x^3$  and  $I(G_2; x) = 1 + 5x + 4x^2$ , we see that both  $G_1$  and  $G_2$  satisfy Proposition 2.1(ii). On the other hand,  $K_{1,3}$  does not agree with Proposition 2.1(ii), because  $\alpha(K_{1,3}) = 3$ ,  $I(K_{1,3}; x) = 1 + 4x + 3x^2 + x^3$ , while  $s_1 = 4 > 3 = s_2$ .

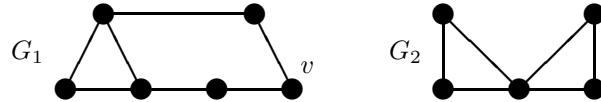


Figure 1: Non-well-covered graph.

**Corollary 2.2** If  $G$  is a well-covered graph with  $\alpha(G) = \alpha$ , then  $s_k \leq s_{\alpha-k}$  for  $0 \leq k \leq \alpha/2$ .

**Proof.** Let  $k \in \{0, \dots, \lfloor \alpha/2 \rfloor\}$ . According to Proposition 2.1(i), we obtain successively, that:

$$\begin{aligned} (\alpha - k) \cdot s_k &\leq (k + 1) \cdot s_{k+1}, \\ (\alpha - k - 1) \cdot s_{k+1} &\leq (k + 2) \cdot s_{k+2}, \\ &\dots \\ (k + 1) \cdot s_{\alpha-k-1} &\leq (\alpha - k) \cdot s_{\alpha-k}. \end{aligned}$$

By multiplying these inequalities, we get

$$\begin{aligned} & (\alpha - k)(\alpha - k - 1)(\alpha - k - 2) \cdots (k + 1) \cdot s_k \cdot s_{k+1} \cdot s_{k+2} \cdots s_{\alpha-k-1} \\ & \leq (k + 1)(k + 2)(k + 3) \cdots (\alpha - k) \cdot s_{k+1} \cdot s_{k+2} \cdot s_{k+3} \cdots s_{\alpha-k-1} \cdot s_{\alpha-k}, \end{aligned}$$

which clearly leads to  $s_k \leq s_{\alpha-k}$ . ■

The above Corollary 2.2 fails for some non-well-covered graphs, e.g.,  $K_{1,3}$  has  $\alpha(K_{1,3}) = 3$ , while  $s_1 = 4 > 3 = s_2 = s_{\alpha-1}$ . Nevertheless,  $P_5$  is not a well-covered graph, but  $I(P_5; x) = 1 + 5x + 6x^2 + x^3$  shows that  $P_5$  satisfies Corollary 2.2.

For a graph  $G$  of order  $n$  and having  $\alpha(G) = \alpha$ , we denote

$$\omega_{\alpha-k} = \max\{n - |N[S]| : S \text{ is a stable set with } |S| = k\}, 0 \leq k \leq \alpha.$$

Clearly,  $\omega_0 = 0, \omega_\alpha = n$ . While  $\omega_1(G) \leq \omega(G)$ , it is not necessary that  $\omega_1(G) = \omega(G)$ . For instance, the graph  $K_3^*$  (depicted in Figure 2) has  $\omega_1 = 2, \omega(K_3^*) = 3$ . It is worth mentioning that for any odd chordless cycle  $C_{2n+1}, n \geq 2$ , or even chordless path  $P_{2n}, n \geq 2$ , these two parameters are identical.

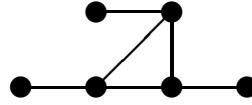


Figure 2: The graph  $K_3^*$ .

**Lemma 2.3** *If  $G$  is a graph of order  $n \geq 1$  with  $\alpha(G) = \alpha$ , then*

$$(k + 1) \cdot s_{k+1} \leq \omega_{\alpha-k} \cdot s_k, 0 \leq k < \alpha,$$

*in particular,  $\alpha \cdot s_\alpha \leq \omega_1 \cdot s_{\alpha-1} \leq \omega(G) \cdot s_{\alpha-1}$ .*

**Proof.** Let  $H = (\mathcal{A}, \mathcal{B}, \mathcal{W})$  be the bipartite graph defined as follows:  $X \in \mathcal{A} \Leftrightarrow X$  is a stable set in  $G$  of size  $k$ , then  $Y \in \mathcal{B} \Leftrightarrow Y$  is a stable set in  $G$  of size  $k + 1$ , and  $XY \in \mathcal{W} \Leftrightarrow X \subset Y$  in  $G$ . Since any  $Y \in \mathcal{B}$  has exactly  $k + 1$  subsets of size  $k$ , it follows that  $|\mathcal{W}| = (k + 1) \cdot s_{k+1}$ . On the other hand, if  $X \in \mathcal{A}$  and, then  $X \cup \{v\} \in \mathcal{B}$  for any  $v \in V(G) - N[X]$ , i.e.,  $X$  has at most  $\omega_{\alpha-k}$  neighbors in  $\mathcal{B}$ . Hence, we get that  $(k + 1) \cdot s_{k+1} = |\mathcal{W}| \leq \omega_{\alpha-k} \cdot |\mathcal{A}| = \omega_{\alpha-k} \cdot s_k$ . In particular, for  $k = \alpha - 1$ , we obtain  $\alpha \cdot s_\alpha \leq \omega_1 \cdot s_{\alpha-1} \leq \omega(G) \cdot s_{\alpha-1}$ . ■

Let us remark that there are quasi-regularizable graphs with non-unimodal independence polynomials, e.g.,

(a)  $G = K_{10} + \sqcup 6K_1$  is connected and has

$$I(G; x) = (1 + x)^6 + 10x = 1 + \mathbf{16}x + 15x^2 + \mathbf{20}x^3 + 15x^4 + 6x^5 + x^6;$$

(b)  $G = (K_{24} + \sqcup 6K_1) \sqcup (K_{25} + \sqcup 6K_1)$  is disconnected and has

$$\begin{aligned} I(G; x) &= \left( (1 + x)^6 + 24x \right) \left( (1 + x)^6 + 25x \right) \\ &= 1 + 61x + \mathbf{960}x^2 + 955x^3 + \mathbf{1475}x^4 + 1527x^5 \\ &\quad + 1218x^6 + 841x^7 + 495x^8 + 220x^9 + 66x^{10} + 12x^{11} + x^{12}. \end{aligned}$$

**Proposition 2.4** *If  $G$  is a quasi-regularizable graph of order  $n = 2\alpha(G) = 2\alpha$ , then*

- (i)  $\omega_{\alpha-k} \leq 2(\alpha - k)$ ,  $0 \leq k \leq \alpha$ ;
- (ii)  $(k+1) \cdot s_{k+1} \leq 2(\alpha - k) \cdot s_k$ ,  $0 \leq k < \alpha$ ;
- (iii)  $s_{\lceil(2\alpha-1)/3\rceil} \geq \dots \geq s_{\alpha-1} \geq s_\alpha$ .

**Proof.** (i) Let  $S$  be a stable set in  $G$  of size  $k \geq 0$ . According to Theorem 1.1, it follows that  $|S| \leq |N(S)|$ , which implies  $2 \cdot |S| \leq |S \cup N(S)| = |N[S]|$  and, hence,  $2 \cdot (\alpha - k) = 2 \cdot (\alpha - |S|) \geq n - |N[S]|$ , because  $n = 2\alpha$ . Consequently, we infer that  $\omega_{\alpha-k} \leq 2(\alpha - k)$ .

(ii) The result follows by combining Lemma 2.3 and part (i).

(iii) The fact that  $(k+1) \cdot s_{k+1} \leq 2(\alpha - k) \cdot s_k$  implies that  $s_{k+1} \leq s_k$  holds for  $k+1 \geq 2(\alpha - k)$ , i.e., for  $k \geq (2\alpha - 1)/3$ .  $\blacksquare$

There are no quasi-regularizable graphs  $G$  of order  $n > 2\alpha(G)$  that satisfy Proposition 2.4(i),(ii), since for  $k = 0$ , each of them demands  $n \leq 2\alpha(G)$ .

In addition, for the graphs  $G_1, G_2$  in Figure 3,  $I(G_1; x) = 1 + 6x + 8x^2$  and  $I(G_2; x) = 1 + 8x + 19x^2 + 12x^3$  show that Proposition 2.4(iii) is sometimes, but not always, valid for a quasi-regularizable graph  $G$  on  $n > 2\alpha(G)$  vertices. Notice that  $G_1$  is also well-covered, but not very well-covered.

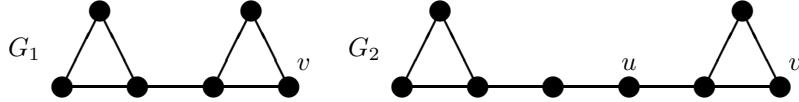


Figure 3:  $G_1, G_2$  are quasi-regularizable graphs, but only  $G_1$  is well-covered.

The graph  $G$  in Figure 4 is very well-covered and its independence polynomial  $I(G; x) = 1 + 12x + 52x^2 + 110x^3 + 123x^4 + 70x^5 + 16x^6$  is not only unimodal but log-concave as well.

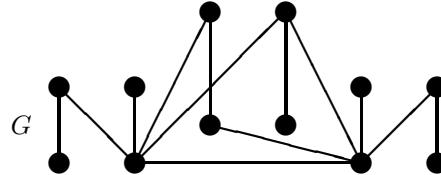


Figure 4: A very well-covered graph with a log-concave independence polynomial.

**Theorem 2.5** *If  $G$  is a very well-covered graph of order  $n \geq 2$  with  $\alpha(G) = \alpha$ , then*

- (i)  $(\alpha - k) \cdot s_k \leq (k+1) \cdot s_{k+1} \leq 2(\alpha - k) \cdot s_k$ ,  $0 \leq k < \alpha$ ;
- (ii)  $s_0 \leq s_1 \leq \dots \leq s_{\lceil\alpha/2\rceil}$  and  $s_{\lceil(2\alpha-1)/3\rceil} \geq \dots \geq s_{\alpha-1} \geq s_\alpha$ ;
- (iii)  $s_{\alpha-2} \cdot s_\alpha \leq s_{\alpha-1}^2$ , where  $\alpha \geq 2$ ;
- (iv)  $I(G; x)$  is unimodal, while  $\alpha \leq 9$ ;
- (v)  $I(G; x)$  is log-concave, while  $\alpha \leq 5$ .

**Proof.** (i) It follows from Proposition 2.1(i) and Proposition 2.4, because any well-covered graph without isolated vertices is quasi-regularizable (see Berge, [3], [4]).

(ii) The assertion is established according to Proposition 2.1(ii) and Proposition 2.4.

(iii) Taking  $k = \alpha - 2$  in Proposition 2.1(i), we get  $2 \cdot s_{\alpha-2} \leq (\alpha - 1) \cdot s_{\alpha-1}$ , while substituting  $k = \alpha - 1$  in part (i) assures that  $\alpha \cdot s_\alpha \leq 2 \cdot s_{\alpha-1}$ , which together lead to  $2\alpha \cdot s_{\alpha-2} \cdot s_\alpha \leq 2(\alpha - 1) \cdot s_{\alpha-1}^2$  and, hence,  $s_{\alpha-2} \cdot s_\alpha \leq s_{\alpha-1}^2$ .

(iv) By part (ii),  $s_0 \leq s_1 \leq \dots \leq s_{\lceil \alpha/2 \rceil}$  and  $s_{\lceil (2\alpha-1)/3 \rceil} \geq \dots \geq s_{\alpha-1} \geq s_\alpha$ . In addition, the fact that  $\alpha(G) \leq 9$  ensures that either  $|\lceil \alpha/2 \rceil - \lceil (2\alpha-1)/3 \rceil| \leq 1$ .

(v) Notice that  $s_0 \cdot s_2 = |E(\overline{G})| \leq |V(G)|^2 = s_1^2$  is true for any graph  $G$  with  $\alpha(G) = \alpha \geq 2$ . By part (iii),  $s_{\alpha-2} \cdot s_\alpha \leq s_{\alpha-1}^2$ . Therefore, we have to check that  $s_{k-1} \cdot s_{k+1} \leq s_k^2$  only for  $k \in \{2, \alpha - 2\}$ .

Part (i) implies that  $(\alpha - k + 1) \cdot s_{k-1} \leq k \cdot s_k$  and  $(k + 1) \cdot s_{k+1} \leq 2(\alpha - k) \cdot s_k$ , which together give

$$(\alpha - k + 1) \cdot (k + 1) \cdot s_{k-1} s_{k+1} \leq 2(\alpha - k) \cdot k \cdot s_k^2.$$

If  $(\alpha - k + 1) \cdot (k + 1) \geq 2(\alpha - k) \cdot k$ , then  $s_{k-1} \cdot s_{k+1} \leq s_k^2$ . In other words, we are interested to know when  $k^2 - \alpha k + \alpha + 1 \geq 0$ , while  $2 \leq k \leq \alpha - 2$ . Since the roots of  $k^2 - \alpha k + \alpha + 1$  are  $k_{1,2} = (\alpha \pm \sqrt{\alpha^2 - 4\alpha - 4})/2$ , we conclude the following, depending on  $\alpha$ :

(a)  $\alpha \leq 4$ , then  $\alpha^2 - 4\alpha - 4 < 0$  and, hence,  $k^2 - \alpha k + \alpha + 1 \geq 0$  is valid for any  $k$ ;

(b)  $\alpha = 5$ , then  $k_1 = 2, k_2 = 3$ , and  $k^2 - \alpha k + \alpha + 1 \geq 0$  is still true for any  $k$ ;

(c)  $\alpha \geq 6$ , then  $k^2 - \alpha k + \alpha + 1 \geq 0$  only for  $k = 1$  and  $k = \alpha - 1$ , because  $2 < (\alpha - \sqrt{\alpha^2 - 4\alpha - 4})/2 < 4$  and  $2(\alpha - 2) < (\alpha + \sqrt{\alpha^2 - 4\alpha - 4})/2 < 2(\alpha - 1)$ .

In summary, the log-concavity condition  $s_{k-1} \cdot s_{k+1} \leq s_k^2, 1 \leq k \leq \alpha - 1$ , holds for  $\alpha \leq 5$ . ■

A graph  $G$  is called *perfect* if  $\chi(H) = \omega(H)$  for any induced subgraph  $H$  of  $G$ , where  $\chi(H)$  denotes the chromatic number of  $H$  (Berge, [2]). Lovász proved the theorem claiming that a graph  $G$  is perfect if and only if  $|V(H)| \leq \alpha(H) \cdot \omega(H)$  for any induced subgraph  $H$  of  $G$  (see [20]).

**Proposition 2.6** *If  $G$  is a perfect graph with  $\alpha(G) = \alpha$  and  $\omega = \omega(G)$ , then  $s_{\lceil (\omega\alpha-1)/(\omega+1) \rceil} \geq \dots \geq s_{\alpha-1} \geq s_\alpha$ .*

**Proof.** Let  $S$  be a stable set in  $G$  of size  $k \geq 0$ . Then  $H = G - N[S]$  is an induced subgraph of  $G$  and has  $\alpha(H) \leq \alpha - k$ . Therefore, by Lovász's theorem,

$$|V(H)| \leq \omega(H) \cdot \alpha(H) \leq \omega(H) \cdot (\alpha - k) \leq \omega \cdot (\alpha - k)$$

and, hence,  $\omega_{\alpha-k} \leq \omega \cdot (\alpha - k)$ . Further, according to Lemma 2.3, we obtain that  $(k + 1) \cdot s_{k+1} \leq \omega \cdot (\alpha - k) \cdot s_k, 0 \leq k < \alpha$ . Now,  $s_{k+1} \leq s_k$  is true while  $k + 1 \geq \omega \cdot (\alpha - k)$ , i.e., for  $k \geq (\omega\alpha - 1)/(\omega + 1)$ . ■

In fact, in Proposition 2.6 there is some  $k$  such that  $\lceil(\omega\alpha - 1)/(\omega + 1)\rceil \leq k < \alpha$  if and only if  $\alpha - \frac{1+\alpha}{1+\omega} \leq \alpha - 1$ , i.e., for  $\alpha \geq \omega$ . It is worth mentioning that, for general graphs, Lemma 2.3 assures that if a graph  $G$  satisfies  $\omega(G) \leq \alpha = \alpha(G)$ , then  $s_\alpha \leq s_{\alpha-1}$ . However, the inverse assertion is not true, e.g.,  $\alpha(K_4 - e) = 2 < 3 = \omega(K_4 - e)$  and  $I(K_4 - e; x) = 1 + 4x + x^2$ , where by  $K_4 - e$  we mean the graph obtained from  $K_4$  by deleting one of its edges.

For non-perfect graphs, Proposition 2.6 is not necessarily false, for example,  $I(C_7; x) = 1 + 7x + 14x^2 + 7x^3$ . However, the graph  $G = \sqcup 4C_5$  is not perfect,  $\alpha(G) = 8, \omega(G) = 2$  and

$$\begin{aligned} I(\sqcup 4C_5; x) &= (1 + 5x + 5x^2)^4 = 1 + 20x + 170x^2 + 800x^3 + 2275x^4 + \\ &\quad + 4000x^5 + \mathbf{4250}x^6 + 2500x^7 + 625x^8 \end{aligned}$$

is log-concave, but it does not satisfy Proposition 2.6, since  $\lceil(\omega\alpha - 1)/(\omega + 1)\rceil = \lceil(2 \cdot 8 - 1)/(2 + 1)\rceil = 5$  and  $s_5 = 4000 < 4250 = s_6$ .

Any minimal imperfect graph  $G$ , i.e.,  $G = C_{2n+1}, n \geq 2$ , or  $G = \overline{C_{2n+1}}, n \geq 2$ , is claw-free and, consequently, its  $I(G; x)$  is log-concave, by Hamidoune's theorem, [11]. However, there are non-perfect graphs, whose independence polynomials are not unimodal, e.g., the disconnected graph  $G = (K_{95} + \sqcup 4K_3) \sqcup C_5$  has

$$\begin{aligned} I(G; x) &= (1 + 107x + 54x^2 + 108x^3 + 81x^4)(1 + 5x + 5x^2) \\ &= 1 + 112x + 594x^2 + \mathbf{913}x^3 + 891x^4 + \mathbf{945}x^5 + 405x^6. \end{aligned}$$

Let  $H = K_{97} + \sqcup 4K_3$ , and  $G$  be the graph obtained from  $H$  by adding an edge from a vertex of  $K_{97}$  to a vertex of some  $C_5$ . Then  $G$  is a connected imperfect graph whose  $I(G; x)$  is not unimodal, since

$$\begin{aligned} I(G; x) &= (1 + 109x + 54x^2 + 108x^3 + 81x^4)(1 + 4x + 3x^2) \\ &\quad + x(1 + 2x)(1 + 108x + 54x^2 + 108x^3 + 81x^4) \\ &= 1 + 114x + 603x^2 + \mathbf{921}x^3 + 891x^4 + \mathbf{945}x^5 + 405x^6. \end{aligned}$$

Since any bipartite graph  $G$  is perfect and has  $\omega(G) \leq 2$ , we obtain the following result.

**Corollary 2.7** *If  $G$  is a bipartite graph with  $\alpha(G) = \alpha \geq 1$ , then  $s_{\lceil(2\alpha-1)/3\rceil} \geq \dots \geq s_{\alpha-1} \geq s_\alpha$ .*

In particular, we infer a similar result for trees, whose importance is significant vis-à-vis the conjecture of Alavi *et al.*

**Corollary 2.8** *If  $T$  is a tree with  $\alpha(T) = \alpha$ , then  $s_{\lceil(2\alpha-1)/3\rceil} \geq \dots \geq s_{\alpha-1} \geq s_\alpha$ .*

For non-bipartite graphs, Corollary 2.7 is not necessarily false (see the graphs in Figure 3).

### 3 Conclusions

In this paper we prove that for very well-covered graphs the "chaotic interval"  $(s_{\lceil \alpha/2 \rceil}, s_{\lceil \alpha/2 \rceil+1}, \dots, s_\alpha)$  involved in the roller-coaster conjecture of Michael and Traves can be shorten to  $(s_{\lceil \alpha/2 \rceil}, s_{\lceil \alpha/2 \rceil+1}, \dots, s_{\lceil (2\alpha-1)/3 \rceil})$ . It seems that one can get even deeper results, by using more efficiently the power of the new defined parameter  $\omega_k$ .

We also conclude with the two following conjectures sharpening the conjectures of Brown *et al.* and Alavi *et al.* respectively.

**Conjecture 3.1**  $I(G; x)$  is log-concave for any very well-covered graph  $G$ .

**Conjecture 3.2**  $I(T; x)$  is log-concave for any (well-covered) tree  $T$ .

### References

- [1] Y. Alavi, P. J. Malde, A. J. Schwenk, P. Erdős, *The vertex independence sequence of a graph is not constrained*, Congressus Numerantium **58** (1987) 15-23.
- [2] C. Berge, *Färbung von Graphen deren sämtliche bzw. deren ungerade Kreise starr sind (Zusammenfassung)*, Wiss.Z. Martin-Luther-Univ. Halle **10** (1961) 114-115.
- [3] C. Berge, *Some common properties for regularizable graphs, edge-critical graphs and B-graphs*, in: Graph Theory and Algorithms Lecture Notes in Computer Science **108** (1980) 108-123, Springer-Verlag, Berlin.
- [4] C. Berge, *Some common properties for regularizable graphs, edge-critical graphs and B-graphs*, Annals of Discrete Mathematics **12** (1982) 31-44.
- [5] J. I. Brown, K. Dilcher, R. J. Nowakowski, *Roots of independence polynomials of well-covered graphs*, Journal of Algebraic Combinatorics **11** (2000) 197-210.
- [6] J. I. Brown, C. A. Hickman, R. J. Nowakowski, *On the location of roots of independence polynomials*, Journal of Algebraic Combinatorics **19** (2004) 273-282.
- [7] R. Dutton, N. Chandrasekharan, R. Brigham, *On the number of independent sets of nodes in a tree*, Fibonacci Quarterly **31** (1993) 98-104.
- [8] O. Favaron, *Very well-covered graphs*, Discrete Mathematics **42** (1982) 177-187.
- [9] A. Finbow, B. Hartnell and R. J. Nowakowski, *A characterization of well-covered graphs of girth 5 or greater*, Journal of Combinatorial Theory B **57** (1993) 44-68.

- [10] I. Gutman, F. Harary, *Generalizations of the matching polynomial*, *Utilitas Mathematica* **24** (1983) 97-106.
- [11] Y. O. Hamidoune, *On the number of independent  $k$ -sets in a claw-free graph*, *Journal of Combinatorial Theory B* **50** (1990) 241-244.
- [12] C. Hoede, X. Li, *Clique polynomials and independent set polynomials of graphs*, *Discrete Mathematics* **125** (1994) 219-228.
- [13] V. E. Levit, E. Mandrescu, *Well-covered and König-Egerváry graphs*, *Congressus Numerantium* **130** (1998) 209-218.
- [14] V. E. Levit, E. Mandrescu, *Well-covered trees*, *Congressus Numerantium* **139** (1999) 101-112.
- [15] V. E. Levit, E. Mandrescu, *On well-covered trees with unimodal independence polynomials*, *Congressus Numerantium Congressus Numerantium* **159** (2002) 193-202.
- [16] V. E. Levit, E. Mandrescu, *On unimodality of independence polynomials of some well-covered trees*, *DMTCS 2003* (C. S. Calude et al. eds.), LNCS **2731**, Springer-Verlag (2003) 237-256.
- [17] V. E. Levit, E. Mandrescu, *A Family of Well-Covered Graphs with Unimodal Independence Polynomials*, *Congressus Numerantium* **165** (2003) 195-207.
- [18] V. E. Levit, E. Mandrescu, *On the Roots of Independence Polynomials of Almost All Very Well-Covered Graphs*, Los Alamos Archive, prE-print arXiv:math. CO/0305227, 2003, 17 pp.
- [19] V. E. Levit, E. Mandrescu, *Independence polynomials of well-covered graphs: generic counterexamples for the unimodality conjecture*, Los Alamos Archive, prE-print arXiv:math. CO/0309151, 2003, 10 pp.
- [20] L. Lovász, *A characterization of perfect graphs*, *Journal of Combinatorial Theory Series B* **13** (1972) 95-98.
- [21] T. S. Michael, W. N. Traves, *Independence sequences of well-covered graphs: non-unimodality and the Roller-Coaster conjecture*, *Graphs and Combinatorics* **19** (2003) 403-411.
- [22] M. D. Plummer, *Some covering concepts in graphs*, *Journal of Combinatorial Theory* **8** (1970) 91-98.
- [23] G. Ravindra, *Well-covered graphs*, *J. Combin. Inform. System Sci.* **2** (1977) 20-21.
- [24] A. A. Zykov, *On some properties of linear complexes*, *Math. Sb.* **24** (1949) 163-188 (in Russian).
- [25] A. A. Zykov, *Fundamentals of graph theory*, BCS Associates, Moscow, 1990.